Dependence of Fourier amplitude spectra at low frequencies on zero pads

Notes by Dave Boore

These notes were prompted by Sinan Akkar's question in September, 2008, as to why the Fourier acceleration spectrum (FAS) from low-cut filtered time series usually show a leveling off toward low frequencies, rather than a continual decrease. Part of the answer almost certainly involves the finite-precision arithmetic in a computer calculation (it is beyond my knowledge of numerical analysis to assess the importance of this effect). But another part of the answer, determined empirically by running `blpadflt` for a number of cases, involves the effect of the length of zero pads (see Boore, 2005, for a discussion of the need to add zeros when doing acausal filtering). In short, the number of zeros given by the standard formula is not of sufficient duration to include enough of the filter transient to fully capture the effect of the low-cut filtering. The standard formula is

$$T_{ZTotal} = 3 \times nroll / f_C$$

(Converse and Brady, 1992, p. 2-3, where the equation above gives the total duration of pads, which is split equally into leading and trailing pads). In the current version of `blpadflt` the filter rolloff is given by $nslope$, which is the power of frequency for the low-cut filter response at low frequencies (i.e., for a bidirectional Butterworth filter, the filter response goes as $(f/f_C)^{nslope}$ at low frequencies). For a bidirectional filter,

$$nroll = nslope/4 .$$

What is the relation between *nslope* and the order *n* of the Butterworth filter? Each pass of an *n*th order Butterworth low-cut filter produces a filter response that goes as $(f/f_C)^{n}$ at low frequencies. Thus the two passes of an *n*th order Butterworth required to produce a zero-phase (acausal) response will have a filter transfer function that goes as $(f/f_C)^{2n}$ at low frequencies. Thus

$$n = 2nroll = nslope/2 .$$

To summarize, here are the equations for the total length of zero pads for an acausal low-cut filter in terms of *nroll*, *n*, and *nslope*. I've written the equations in terms of the period T_C of the low-cut filter, as it is easier for me to think of the length of the zero pads in terms of a multiple of the period cutoff of the filter

$$T_{ZTotal} = 3 \times nroll \times T_C$$

$$T_{ZTotal} = \frac{3}{2} \times n \times T_C$$

$$T_{ZTotal} = \frac{3}{4} \times nslope \times T_C$$

For example, for a 0.1 Hz Butterworth bidirectional (acausal) filter, $nslope = 8$ low-cut filter $T_{ZTotal} = 60$ s. The one example I show in this note suggests that increasing the duration by a factor of two will capture as much of the filter transient as is necessary, before the FAS for low-frequency filtered time series runs up against the finite-precision problem.

Another thing found in this investigation is that the FAS of a causal filter of a non-padded time series can be a poor approximation of the actual FAS at low frequencies. Padding with trailing zeros is required, but this is not an option in `blpadflt` because the peak motions are not affected by the filter transient, although the response spectrum can be.

Here is a plot of the acceleration time series for the 1999 Duzce, Turkey, record 19991112.165720.EW.IRIGM496.DUZCE.SMC from station ???.

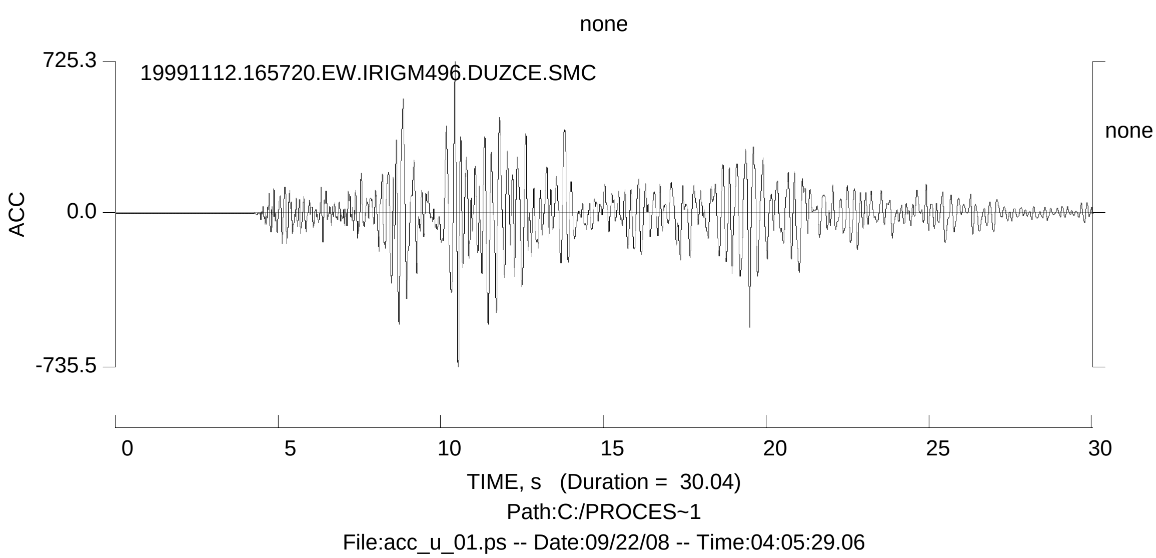


Figure 1.

And here is a plot of the FAS for a series of filters applied to the acceleration time series:

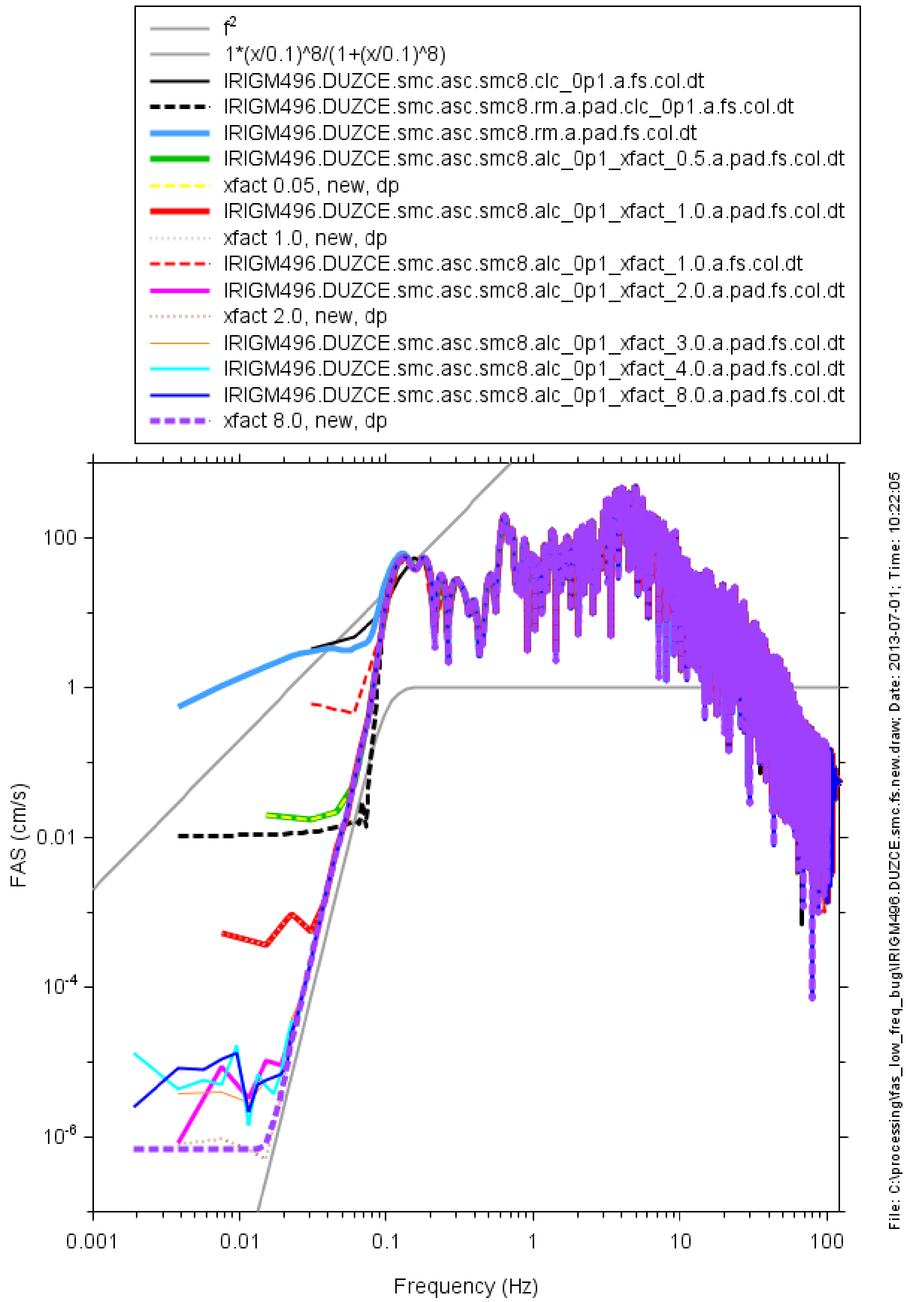
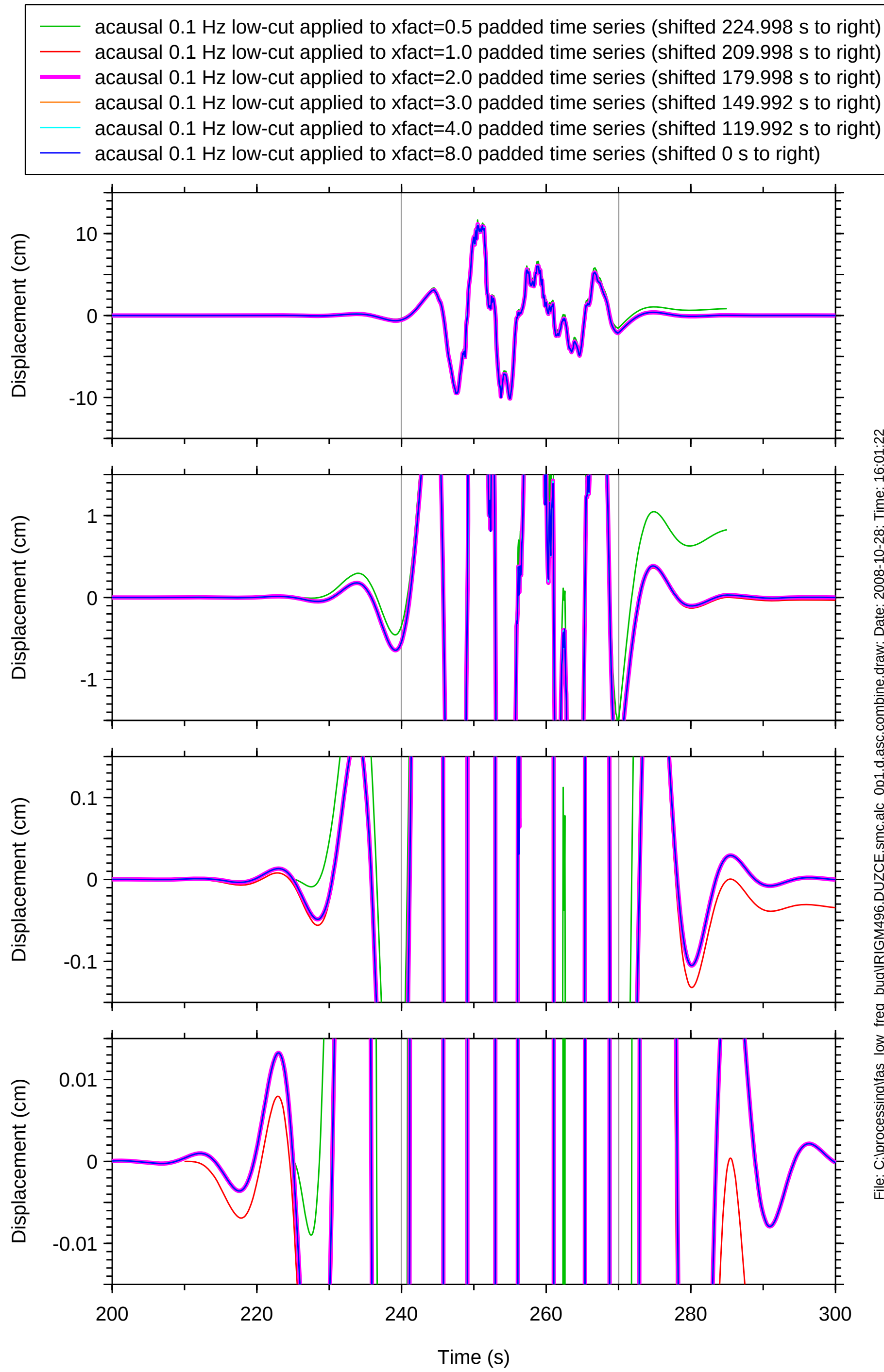


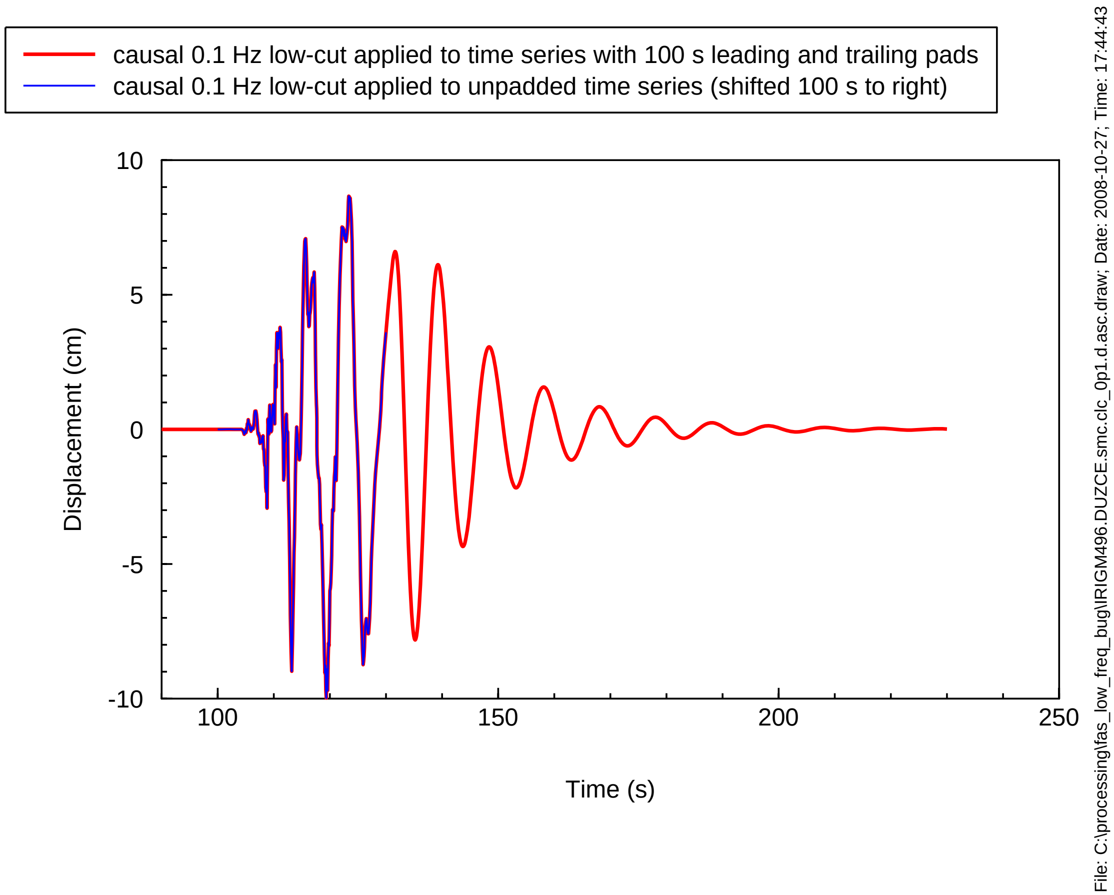
Figure 2.

The two black curves are for the causally filtered record, to be discussed later. The thick light blue curve is the FAS of the unfiltered record, after removing the mean and appending 100 s of leading and trailing zeros (in order to get estimates of the FAS at lower frequencies). The upper gray line was added by eye and has a slope of 2 in log-log scaling; it represents a functional dependence going as f^2; this is the functional dependence for a point dislocation source at far-field, far-source distances and is often used to judge the frequency at which low-frequency noise exceeds the signal. The functional dependence for near-source recordings such as this one, however, is probably different than f^2. The other gray line shows the theoretical filter response for the acausal filter used in this example (this filter response does not apply to the causal filter results shown by the black lines). The rest of the solid curves show the FAS for time series filtered after appending various durations of leading and trailing zeros, as given by "xfact". $xfact = 1.0$ is the duration from the standard formula, $xfact = 3.0$ is three times that duration, etc. The dashed red curve shows the result of computing the FAS after stripping off the zero-pad portions of the record (for $xfact = 1.0$). Finally, results are shown from a program that uses double precision in computing the Fourier transform (these are labeld "dp" in the figure). This graph tells the story: A lot is gained in going from no added zeros to $xfact = 2.0$, but essentially little is gained after that (the plateau at low frequencies presumably being due to numerical finite-precision effects). The low-frequency plateau for FAS from no zero pad or the standard pad (xfact=1.0) are the same for FAS computed without and with double precision; it is only for the FAS from longer pads that the precision of the FAS computations shows up (see the xfact 2.0 and 8.0 curves). So whether or not the FAS is computed using double precision is not an issue. The ability of the added zeros to capture the filter transients is shown in the next plot, where a series of enlargements of the y-axis are shown to see better the very low-level transients.

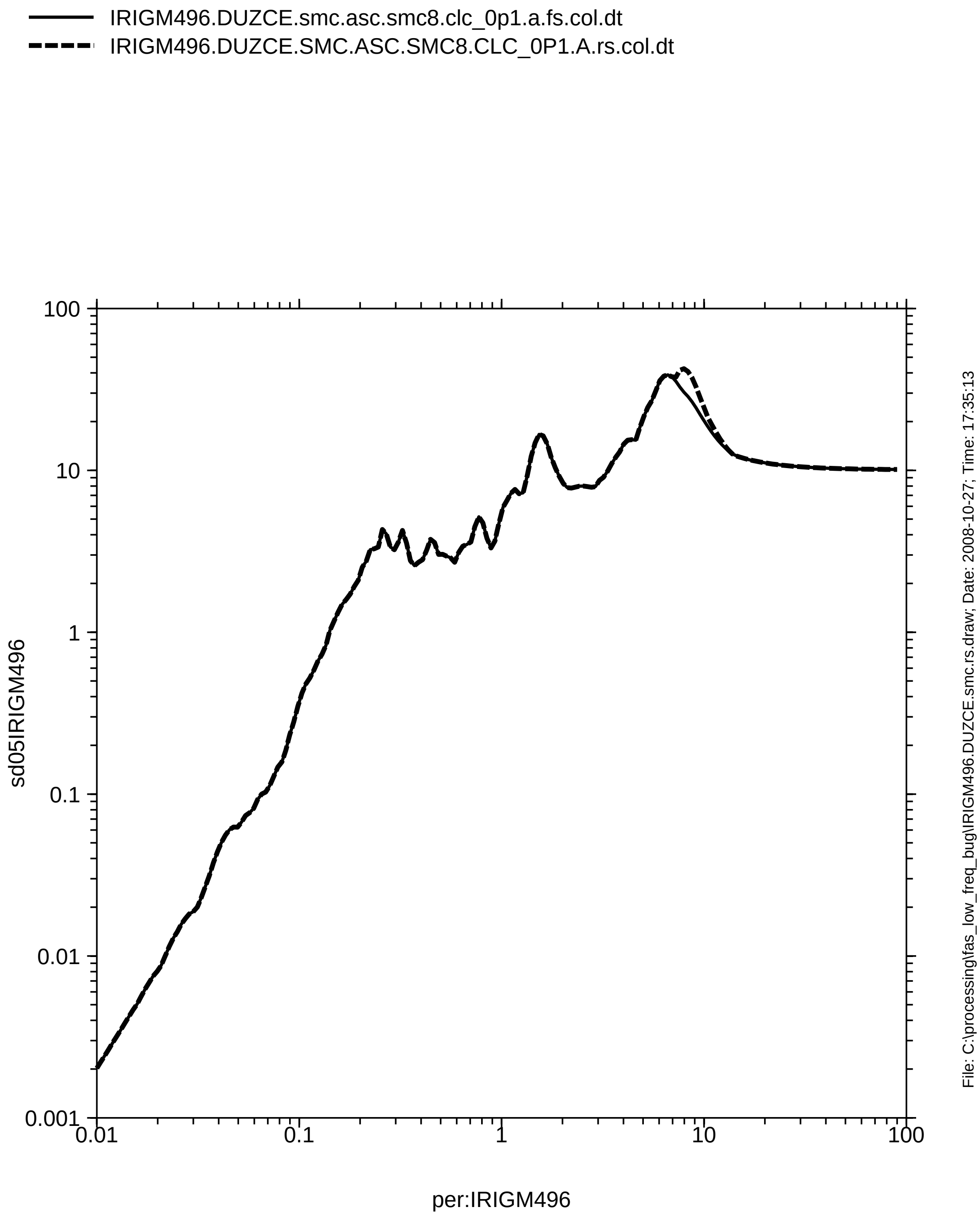


In these graphs the curves for $xfact \geq 2.0$ are on top of one another. The vertical gray lines indicate the duration of the original time series. The horizontal is the same for all graphs. As best seen in the lower two graphs, there are small differences between the standard case and the cases with a longer duration of added pads; these small differences show up in the FAS at low frequencies.

Turning now to causal filters, there is a large difference in the low-frequency FAS for time series with and without zeros appended to the data before filtering.   The reason can be seen in the following plot of the displacement time series of the filtered records:



Clearly the peak displacement will be the same, but the unpadded time series will be missing a large amount of the filter transient. So even though there will be no difference in the portion of the time series corresponding to the original data, the FAS will be very different.   But this being so, what about the response spectrum? A shown here



There will be a difference in the response spectra, although they must be the same at low and high periods (corresponding to PGA and PGD, respectively). This example suggests that `blpadflt` should be modified to add trailing zeros before causal filtering, but as we advocate acausal rather than causal filters, it may be awhile before I include that modification.

Future Work

Redo the computations with other time series (Chris Stephens pointed out that the one used here does not show as much “noise” at low frequencies for the FAS from the unfiltered record as in many records---although the FS does decay less rapidly than f^2, as shown by the gray line in the FAS plot, which may or may not be an indication of noise---, and in this regard might be anomalous).

References

Boore, D. M. (2005). On pads and filters: Processing strong-motion data, *Bull. Seism. Soc. Am.* **95**, 745—750.

Converse, A.M. and A.G. Brady (1992). BAP --- Basic strong-motion accelerogram processing software; Version 1.0, *U. S. Geological Survey Open-File Report 92-296A*, 174p.